

# An Introduction to Vectors

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## THE LAWS OF MOTION

The three laws that govern motion are usually attributed to Isaac Newton. They are:

1. Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.
2. Rate of change motion is proportional to the impressed force, and is in the direction in which the force acts.
3. To every action there is always an equal an opposite reaction.

The bodies referred to in the First Law should be taken to be particles. Particles are idealizations of objects that have no size. The Laws of Motion can, in most cases, be generalized to systems of interacting particles, although care must be taken in some specific instances.

The tendency of a particle to continue in a state of rest or uniform motion is called *inertia*. *Mass* is the quantitative measure of inertia of a body and it is given numerical value by comparison to a standard. An *inertial reference frame* is one in which Newton's First Law holds strictly. Such a frame is also an idealization rather than a physically realizable frame.

Changes in position are measured in terms of distances and times. Distances and times are given quantitative meaning by comparison to standards.

The concept of motion in the Second Law can be given quantitative measure as the momentum of a particle. If  $m$  is the mass and  $\mathbf{v}$  is the rate of change of position, then the *momentum*  $\mathbf{p}$  is given by  $\mathbf{p} = m\mathbf{v}$ . Newton's Second Law then can be expressed as a vector equation,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m \frac{d\mathbf{v}}{dt}.$$

Conceptually, forces are pushes or pulls. The fundamental forces of classical mechanics are either the gravitational force or the electromagnetic force or some manifestations (friction, viscosity, contact, etc.) of these which may obscure their true origins. Forces are given quantitative meaning through Newton's Second Law which connects force to quantitative measures of mass, length and time.

Newton's three Laws of Motion are based on experiment and cannot be proved or derived.

## VECTORS

There are two aspects to motion. The first is called *kinematics* and the second is called *dynamics*. Kinematics is purely the description of motion independent of the laws of physics which govern it. The latter study is dynamics. We are particularly interested in the dynamics of particles and systems, but before we turn to dynamics, we have to develop a mathematical language for describing motion.

We will be concerned with two kinds of quantities. The first, called *scalar* quantities, are characterized by magnitude only and are represented by ordinary real numbers. Time and temperature are scalar quantities. Other quantities have the combined characteristics of magnitude and direction. These are represented by mathematical objects called *vectors*. Relative position, velocity, force and acceleration are vector quantities.

In three-dimensional space, you can think of a vector as a directed line segment having length (magnitude) and direction. You may also think of it in terms of an ordered set of three scalar *components*. In texts, vectors are usually represented in bold face.

$$\mathbf{a} = (a_1, a_2, a_3)$$

The essential characteristics of vectors are the following:

### Equality of Vectors

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are equal if and only if their respective components are equal, i.e.  $\mathbf{a} = \mathbf{b}$  means that  $a_1 = b_1$ ,  $a_2 = b_2$  and  $a_3 = b_3$ .

### Vector Addition

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

### Vector Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

### Null Vector

There exists a null vector  $\mathbf{0}$  such that  $\mathbf{a} + \mathbf{0} = \mathbf{a}$ .

## Commutative Law of Addition

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

## Associative Law of Addition

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

In component form this can be written:

$$\begin{aligned} &(a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3)) \\ &= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, (a_3 + b_3) + c_3) \end{aligned}$$

## Multiplication of a Vector by a Scalar

There are several kinds of multiplication defined for vectors. Let  $p$  be a scalar and let  $\mathbf{a}$  be a vector. Then,

$$p\mathbf{a} = p(a_1, a_2, a_3) = (pa_1, pa_2, pa_3)$$

## Scalar Multiplication of Vectors; Dot Product

There are two kinds of multiplication defined for vectors with vectors. The first is called the *dot product*.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = \sum a_i b_i = a_i b_i.$$

Here,  $|\mathbf{a}|$  is the *magnitude* of  $\mathbf{a}$ , i.e.,

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i a_i},$$

and  $\theta$  is the angle between the directions of  $\mathbf{a}$  and  $\mathbf{b}$ . Very often, for brevity, an index such as  $i$  in the expression  $a_i a_i$  that is repeated exactly twice will imply a summation on that index. In such cases the summation sign  $\sum$  is not written and the index  $i$  is said to be a dummy index because when you actually write out the expression, it is replaced by 1s, 2s and 3s. Such is the case for

$$a_i a_i \equiv \sum a_i a_i \equiv a_1^2 + a_2^2 + a_3^2 \equiv a_j a_j.$$

The dot product of two vectors is itself a scalar; hence this form of multiplication is described as “scalar multiplication” of vectors.

The following are theorems for the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$p(\mathbf{a} \cdot \mathbf{b}) = (p\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (p\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})p$$

## Vector Multiplication of Vectors; Cross Product

The second kind of multiplication produces a vector. Again  $\theta$  is the angle between the directions of  $\mathbf{a}$  and  $\mathbf{b}$ . We then have,

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}},$$

where  $\mathbf{n}$  is a vector of unit length and direction perpendicular to the plane which contains both  $\mathbf{a}$  and  $\mathbf{b}$ . The direction is further defined by a *right hand rule*, i.e., the direction of  $\hat{\mathbf{n}}$  is the direction the thumb of our right hand would be pointing if you pointed the fingers of your right hand along the direction of  $\mathbf{a}$  and curled them toward the direction of  $\mathbf{b}$ . In truth, the cross product is not a true vector, but rather is a *pseudovector*. It does have magnitude and components like a vector, but its direction is ambiguous and has to be defined artificially by a right-hand-rule. Physical quantities, such as the magnetic field, that are represented by pseudovectors do not have a true direction, but are given one artificially by a right-hand-rule.

We may also write,

$$c_i = \sum_{j,k} \delta_{ijk} a_j b_k \equiv \delta_{ijk} a_j b_k$$

where, again, the repeated indices  $j$  and  $k$  indicate a sum and are dummies. The symbol  $\delta_{ijk}$  is a  $3 \times 3 \times 3$  matrix called the *Levi-Civita tensor*. If  $i, j$  and  $k$  have the values 1, 2 and 3 respectively or any even permutation of (123), then  $\delta_{ijk} = +1$ . If the values of  $i, j$ , and  $k$  respectively are an odd permutation of (123) then  $\delta_{ijk} = -1$ . Otherwise (such as in the case of two indices being the same),  $\delta_{ijk} = 0$ . (Think of the indices as labeled beads on a loop of string. An *even* permutation is one you can get by simply moving the beads counterclockwise or clockwise along the loop to change their order as they sit in your hand. Thus (312) and (231) are even permutations of (123). An *odd* permutation, on the other hand, is one for which you would have to exchange the positions of two adjacent indices. Thus (213) and (132) are odd permutations of (123).) While this form takes a little getting used to, it can be a powerful way of writing and manipulating vectors. Please note that it is the  $i$ th *component* and not the full pseudovector that is given by the subscript form.

We have the following theorems:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$p(\mathbf{a} \times \mathbf{b}) = (p\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (p\mathbf{b}) = (\mathbf{a} \times \mathbf{b})p$$

## Unit Vectors

We define a special set of vectors that have unit length. Usually (almost always) we will choose three vectors that are mutually orthogonal. Thus,

$$\hat{\mathbf{e}}_1 = (1, 0, 0)$$

$$\hat{\mathbf{e}}_2 = (0, 1, 0)$$

$$\hat{\mathbf{e}}_3 = (0, 0, 1)$$

However, observe that in this instance the subscripts are used to number the three unit vectors rather than specify components of the vectors. With this in mind and given that the three unit vectors are ordered by a right-hand-rule, i.e., that the direction of  $\hat{\mathbf{e}}_3$  is the direction of  $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$ , we have

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$$

where  $\delta_{ij}$  is a matrix whose values are 1 if  $i = j$  but 0 otherwise. Using the convention that a repeated index implies summation on that index, we may also write,

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \delta_{ijk} \hat{\mathbf{e}}_k.$$

An arbitrary vector may be written in terms of unit vectors,

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3.$$

## Derivatives of Vectors

The derivative of a vector with respect to a scalar is defined by:

$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t}.$$

One may easily show that

$$\frac{d}{dt}(p\mathbf{a}) = p \frac{d\mathbf{a}}{dt} + \frac{dp}{dt} \mathbf{a}$$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{b}$$

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}.$$

As long as the direction and length of the unit vectors is unchanging, we may write,

$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt} \hat{\mathbf{e}}_1 + \frac{da_2}{dt} \hat{\mathbf{e}}_2 + \frac{da_3}{dt} \hat{\mathbf{e}}_3 \equiv \frac{da_i}{dt} \hat{\mathbf{e}}_i.$$

## Directional Derivative, $d\psi_{PQ}$

1.  $\psi(x_1, x_2, x_3) = c$  where  $c$  is a constant defines a surface in Cartesian coordinates. If the constant  $c$  is changed by a small amount, a second surface near the first is defined. An example of such a surface would be the sphere defined by

$$x^2 + y^2 + z^2 = a^2.$$

In the example,  $a$  is the radius of the spherical surface. Changing  $a$  creates a nested set of spherical surfaces.

2. Suppose  $P$  and  $Q$  are two points that are on separate surfaces that are close together so that  $P$  and  $Q$  are close together. The *chain rule* of differentiation gives us an estimate of the change in  $\psi$  as one moves from  $P$  to  $Q$ .

$$d\psi_{PQ} = \frac{\partial\psi}{\partial x_1} dx_1 + \frac{\partial\psi}{\partial x_2} dx_2 + \frac{\partial\psi}{\partial x_3} dx_3.$$

Using subscript notation and summation convention, this can be written,

$$d\psi_{PQ} \equiv \partial_i \psi dx_i \equiv \nabla_i \psi dx_i.$$

We may also define a vector  $ds$  which connects  $P$  to  $Q$ ,

$$ds = dx_1 \hat{\mathbf{e}}_1 + dx_2 \hat{\mathbf{e}}_2 + dx_3 \hat{\mathbf{e}}_3 = dx_i \hat{\mathbf{e}}_i$$

and define the *gradient* of  $\psi$  as,

$$\nabla\psi = \frac{\partial\psi}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{\partial\psi}{\partial x_2} \hat{\mathbf{e}}_2 + \frac{\partial\psi}{\partial x_3} \hat{\mathbf{e}}_3 \equiv \nabla_i \psi \hat{\mathbf{e}}_i.$$

Thus,

$$d\psi_{PQ} = \nabla\psi \cdot ds.$$

3. Observe that if  $P$  and  $Q$  were in the *same* surface,  $d\psi_{PQ} = 0$  for a  $ds$  connecting them. This means that the gradient would have to be perpendicular to the surface at each point on the surface.

## Transformation of Vectors

Consider two sets of orthogonal unit vectors  $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$  and  $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$ . The second set is rotated in a completely general way relative to the first so that there is an angle  $\alpha_{11}$  between  $\hat{\mathbf{e}}'_1$  and  $\hat{\mathbf{e}}_1$ , an angle  $\alpha_{12}$  between  $\hat{\mathbf{e}}'_1$  and  $\hat{\mathbf{e}}_2$ , etc. We may use either of these two sets of unit vectors to express an arbitrary vector  $\mathbf{a}$ ,

$$\mathbf{a} \equiv a_i \hat{\mathbf{e}}_i \equiv a'_i \hat{\mathbf{e}}'_i.$$

We may form a dot product of these expressions with one of the unit vectors of the second set to form,

$$a_i (\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i) = a'_i (\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}'_i).$$

But  $\hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}'_i = \delta_{ji}$ . Thus

$$a'_j = \cos(\alpha_{ji})a_i.$$

If we define a matrix  $S$  of the cosines of the angles between the primed and unprimed axes, the elements of the matrix are defined to be  $S_{ij} = \cos(\alpha_{ij})$ , i.e.,

$$a'_j = S_{ji}a_i.$$

Using a similar argument one can also show that

$$a_j = S_{ij}a'_i.$$

The matrix  $S$  is called the *transformation matrix*, the elements of the matrix  $S_{ij}$  are called the *transformation coefficients* and the set of  $\cos \alpha_{ij}$  are called the *direction cosines* of the primed axis set relative to the unprimed set. It is also easily shown that

$$\hat{\mathbf{e}}'_i = S_{ij}\hat{\mathbf{e}}_j,$$

where, again, the repeated index implies summation.

## Index notation for Cartesian vectors

The symbol  $\delta_{ij}$  is called the Kronecker delta. It is a 3x3 matrix whose elements are 1 if  $i = j$  and 0 if  $i \neq j$ . The symbol  $\delta_{ijk}$  is called the Levi-Civita tensor. It is a 3x3x3 matrix whose elements are 1 if (ijk) stand for an even permutation of (123), -1 if (ijk) stand for an odd permutation of (123), and 0 if any two indices are the same. (123) are labels for the axes of a right-handed, Cartesian coordinate system.

Unless otherwise indicated, it is assumed that there is a summation on repeated indices.

$\mathbf{a} \cdot \mathbf{b} = a_i b_i$		(dot product)
$\mathbf{c} = \mathbf{a} \times \mathbf{b}$	$; c_i = \delta_{ijk} a_j b_k$	(cross product)
$\det(a_{ij}) = a_{1i} a_{2j} a_{3k} \delta_{ijk}$		(determinant)
$\det(a_{ij}) = a_{i1} a_{j2} a_{k3} \delta_{ijk}$		(determinant)
$\nabla \cdot \mathbf{a} = \nabla_i a_i$		(divergence)
$\mathbf{c} = \nabla \times \mathbf{a}$	$; c_i = \delta_{ijk} \nabla_j a_k$	(curl)
$\nabla \psi = \hat{\mathbf{e}}_i \nabla_i \psi$		(gradient)
$\nabla^2 \psi = \nabla_i \nabla_i \psi$		(Laplacian)

The rules are:

1. Repeated indices mean “sum”. The repeated index is a dummy and may have *any* symbol as an index that is otherwise unused in a particular term.
2. No more than two identical indices are allowed in any term.
3. The operator  $\nabla_i \equiv \partial/\partial x_i$  operates on everything to its right.

Example:  $\nabla_i a_j b_k = a_j \nabla_i b_k + b_k \nabla_i a_j$

4. The operator  $\nabla_i$  operating on a Cartesian coordinate,  $x_j$  yields 1 if  $i = j$ , but yields 0 if  $i \neq j$ .

$$\begin{aligned} \nabla_i x_j &= \delta_{ij}, \text{ where} \\ \delta_{ij} &= 0 \quad \text{if } i \neq j \\ &= 1 \quad \text{if } i = j \end{aligned} \quad (\text{Kronecker delta})$$

5. Products of Kronecker deltas may be reduced if an index is repeated (summation).

$$\begin{aligned} \delta_{ij} \delta_{kj} &= \delta_{ik} \\ \text{but } \delta_{ij} \delta_{ij} &= \delta_{jj} = 3 \end{aligned} \quad (\text{trace of } \delta_{ij})$$

6. Relationships among elements of the Levi-Civita tensor.

$$\begin{aligned} \delta_{ijk} &= -\delta_{jik} = -\delta_{ikj} = -\delta_{kji} && (\text{odd permutations}) \\ \delta_{ijk} &= \delta_{kij} = \delta_{jki} && (\text{even permutations}) \end{aligned}$$

7. Products of Levi-Civita tensors may be reduced if an index is repeated (summation) with the following pattern:

$$\delta_{ijk} \delta_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

Example:

$$\begin{aligned} (\mathbf{a} \times \nabla) \cdot \mathbf{r} &= \delta_{ijk} a_j \nabla_k x_i \\ &= \delta_{ijk} a_j \delta_{ki} \\ &= \delta_{iji} a_j \\ &= 0 \end{aligned}$$

## POSITION, VELOCITY, ACCELERATION

The position  $\mathbf{r}$  of a particle relative to a chosen origin is a vector quantity. Like any other vector, it can be written,

$$\mathbf{r} = x_i \hat{\mathbf{e}}_i.$$

The magnitude of  $\mathbf{r}$  is a scalar quantity given by  $r = \sqrt{x_i x_i}$ . The time derivative of  $\mathbf{r}$  is the *velocity*  $\mathbf{v}$ ,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} = \frac{d}{dt}(x_i \hat{\mathbf{e}}_i) = \dot{x}_i \hat{\mathbf{e}}_i + x_i \frac{d\hat{\mathbf{e}}_i}{dt}.$$

In a coordinate system that is fixed in space, the unit vectors are constant in magnitude and in direction. In this case the time derivatives of the unit vectors vanish and we may write,

$$\mathbf{v} = \dot{x}_i \hat{\mathbf{e}}_i.$$

The magnitude of the vector  $\mathbf{v}$  is the *speed*  $v$ . In a fixed coordinate system,

$$v = \sqrt{\dot{x}_i \dot{x}_i}.$$

The time derivative of the velocity is the *acceleration*,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} \equiv \ddot{\mathbf{r}}.$$

In a fixed coordinate system,

$$\mathbf{a} = \frac{d\dot{x}_i}{dt} \hat{\mathbf{e}}_i \equiv \ddot{x}_i \hat{\mathbf{e}}_i.$$

Sometimes it is convenient to use a set of unit vectors that is *not* fixed in space. While the unit vectors continue to have constant unit length, their directions may be changing. As a result the unit vectors have time derivatives that do not vanish. There are two circumstances where this commonly occurs.

First, consider a particle moving along a curved path through space. Let  $\hat{\mathbf{t}}$  be a unit vector tangent to the particle's path. As the particle moves from point to point along the curved path,  $\hat{\mathbf{t}}$  changes direction. At two times separated by  $\Delta t$ ,

$$\hat{\mathbf{t}}(t + \Delta t) = \hat{\mathbf{t}}(t) + d\hat{\mathbf{t}}.$$

You can think of the three vectors in this expression forming a small triangle with an angle  $d\theta$  between  $\hat{\mathbf{t}}(t + \Delta t)$  and  $\hat{\mathbf{t}}(t)$ . Then,

$$\mathbf{v} = v\hat{\mathbf{t}}$$

and,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt}.$$

If  $\Delta t$  is very small,  $d\theta$  is very small and  $d\hat{\mathbf{t}}$  is (almost) perpendicular to  $\hat{\mathbf{t}}$ . Let  $\hat{\mathbf{n}}$  be a unit vector along  $d\hat{\mathbf{t}}$  in the limit that  $\Delta t \rightarrow 0$ . The magnitude of  $d\hat{\mathbf{t}}$  is  $(1)d\theta$ . Hence,  $d\hat{\mathbf{t}} = d\theta\hat{\mathbf{n}}$ . Dividing by  $dt$ , we have,

$$\frac{d\hat{\mathbf{t}}}{dt} = \frac{d\theta}{dt}\hat{\mathbf{n}} = \dot{\theta}\hat{\mathbf{n}}$$

and,

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{t}} + v\dot{\theta}\hat{\mathbf{n}}.$$

For  $\Delta t$  small, the particle can be thought of as moving along a path that can be approximated as a short segment of a circle with radius  $\rho$ . Then  $v = \rho\dot{\theta}$  and,

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{\rho}\hat{\mathbf{n}}.$$

In general the *radius of curvature*  $\rho$  will be a function of the position of the particle along its trajectory. The formula is instantly applicable to particles moving in circles for which

$\rho$  is the radius of the circle. We observe that acceleration has a tangential component  $dv/dt$  long its tangent and a *centripetal* component  $v^2/\rho$  towards the center of the arc of its trajectory. Compare this with the form of  $\mathbf{a} = \ddot{x}_i \hat{\mathbf{e}}_i$  in a fixed coordinate system. Both are correct. The two are just different ways of describing the same thing.

Second, consider a frame of reference  $F'$  with a set of unit vectors  $\hat{\mathbf{e}}'_i$ . Assume that the set is rotating in some arbitrary fashion. To characterize the rotation, we will invent a pseudovector which we will call the *angular velocity*  $\omega$ . We will imagine the instantaneous rotation to be about some axis along which we will put a unit vector  $\hat{\lambda}$ . To give a direction to  $\omega$  we will define a positive rotation along  $\hat{\lambda}$  by a right-hand-rule, i.e., the rotation is positive if the thumb of your right hand points along  $\hat{\lambda}$  when your fingers curl forward in the direction of the rotation.

As the result of an infinitesimal rotation of the system of unit vectors, the unit vector  $\hat{\mathbf{e}}_1$  changes direction such that

$$\hat{\mathbf{e}}'_1(t + \Delta t) = \hat{\mathbf{e}}'_1(t) + d\hat{\mathbf{e}}'_1.$$

Since  $\Delta t$  is small,  $d\hat{\mathbf{e}}'_1$  is perpendicular to  $\hat{\mathbf{e}}'_1$  so that we may write,

$$d\hat{\mathbf{e}}'_1 = (ds_2)\hat{\mathbf{e}}'_2 + (ds_3)\hat{\mathbf{e}}'_3$$

where  $ds_2 = (1)d\theta_3$  and  $ds_3 = (1)d\theta_2$ . Dividing by  $dt$ ,

$$\frac{d\hat{\mathbf{e}}'_1}{dt} = \frac{d\theta_3}{dt}\hat{\mathbf{e}}'_2 + \frac{d\theta_2}{dt}\hat{\mathbf{e}}'_3 = \dot{\theta}_3\hat{\mathbf{e}}'_2 + \dot{\theta}_2\hat{\mathbf{e}}'_3.$$

If we now identify the  $\dot{\theta}_s$  to be the components of a pseudovector  $\omega$  in the primed set, then

$$\frac{d\hat{\mathbf{e}}'_1}{dt} = \omega'_3\hat{\mathbf{e}}'_2 - \omega'_2\hat{\mathbf{e}}'_3 = \omega \times \hat{\mathbf{e}}'_1 = \delta_{1jk}\hat{\mathbf{e}}'_j\omega'_k.$$

The negative sign in front of  $\omega'_2$  arises because a right-hand screw twisted in the sense of increasing  $\theta_2$  is a negative rotation about  $\hat{\mathbf{e}}_2$  by the right-hand-rule that gives directions to rotations. We can derive similar expressions for  $\hat{\mathbf{e}}'_2$  and  $\hat{\mathbf{e}}'_3$ . All three such relationships are summarized by,

$$\frac{d\hat{\mathbf{e}}'_i}{dt} = \omega \times \hat{\mathbf{e}}'_i = \delta_{ijk}\hat{\mathbf{e}}'_j\omega'_k.$$

Now consider two frames  $F$  and  $F'$  each with origins  $O$  and  $O'$  and sets of unit vectors  $\hat{\mathbf{e}}_i$  and  $\hat{\mathbf{e}}'_i$  respectively. The unprimed frame is fixed in space and its unit vectors are constant. Assume that  $F'$  is moving relative to  $F$  with arbitrary translation and rotation. Let  $\mathbf{r}$  be the position of a particle relative to  $O$ ,  $\mathbf{r}'$  be the position of the same particle but with respect to  $O'$ . Finally, let  $\mathbf{R}$  be the position of  $O'$  with respect to  $O$ . Then

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}.$$

The vectors  $\mathbf{r}$ ,  $\mathbf{r}'$  and  $\mathbf{R}$  can all be expressed either in terms of  $\hat{\mathbf{e}}_i$  or  $\hat{\mathbf{e}}'_i$  with different sets of components in the two systems related by a transformation. One of the several ways we can write  $\mathbf{r} = \mathbf{r}' + \mathbf{R}$  in terms of unit vectors is,

$$x_i\hat{\mathbf{e}}_i = x'_i\hat{\mathbf{e}}'_i + X_i\hat{\mathbf{e}}_i.$$

Be careful to note that we have chosen to express  $\mathbf{r}'$  in terms of  $\hat{\mathbf{e}}'_i$  but have expressed  $\mathbf{r}$  and  $\mathbf{R}$  in terms of  $\hat{\mathbf{e}}_i$ . We could also have written  $\mathbf{r}' = x_i^* \hat{\mathbf{e}}_i$  where the asterisk is used to indicate components of  $\mathbf{r}'$  in the unprimed system, but it would not have served our purpose as you will see below.

If we differentiate with respect to time,

$$\dot{x}_i \hat{\mathbf{e}}_i = \dot{x}'_i \hat{\mathbf{e}}'_i + x'_i \frac{d\hat{\mathbf{e}}'_i}{dt} + \dot{X}_i \hat{\mathbf{e}}_i.$$

Using,

$$\frac{d\hat{\mathbf{e}}'_i}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}'_i,$$

we have,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_o.$$

If we differentiate again with respect to time,

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}'}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}' + \boldsymbol{\omega} \times \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{V}_o}{dt}.$$

Using,

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}$$

$$\frac{d\mathbf{v}'}{dt} = \frac{d}{dt}(x'_i \hat{\mathbf{e}}'_i) = \mathbf{a}' + \boldsymbol{\omega} \times \mathbf{v}'$$

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega'_i \hat{\mathbf{e}}'_i) = \dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times \boldsymbol{\omega}) \equiv \dot{\boldsymbol{\omega}}$$

$$\frac{d\mathbf{r}'}{dt} = \frac{d}{dt}(x'_i \hat{\mathbf{e}}'_i) = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'$$

$$\frac{d\mathbf{V}_o}{dt} = \mathbf{A}_0,$$

we then have

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0.$$

In this form, we have the acceleration expressed as a series of terms. The first term on the right is the acceleration of the particle relative to  $O'$ , the second is called the *Coriolis term*, the third is the *transverse acceleration*, the fourth is the *centripetal acceleration* and the fifth is the acceleration of  $O'$  relative to  $O$ . We will return to this equation later, but derive it here to emphasize that when the unit vectors are themselves changing with time, care must be taken if motion is to be correctly described in such a system.

## PROBLEMS

1. Let

$$\mathbf{a} = (1, 1, 1),$$

$$\mathbf{b} = (1, 2, 3),$$

and the matrix A,

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Evaluate explicitly

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i,$$

$$c_i = \delta_{ijk} a_j b_k,$$

$$c_i = A_{ij} a_j,$$

$$c_i = A_{ji} a_j,$$

$$A_{ii},$$

$$d = a_i A_{ij} b_j.$$

2. To gain experience in using the summation convention for Cartesian vectors, prove the following:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3. A vector  $\mathbf{a}$  has components  $(1, 1, 1)$  in a Cartesian system. What are its components in a system obtained by a  $+60^\circ$  rotation about the  $x_2$ -axis? Note that the sense of positive rotation is given by a right-hand-rule.