Ordinary Differential Equations

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INTRODUCTION

Newton's Second Law of motion in one dimension, $F = m\ddot{x}$, is a second order differential equation. In the simplest cases or in the lowest order of approximation, this equation will sometimes reduce to a linear equation. Linear equations are the most studied of differential equations and have the most systematic methods for solution. Hence, textbook examples typically center on problems that can be reduced to linear equations.

In those problems where the force can be derived as the derivative of a potential function, $F_x = -dV/dx$, the potential may often be such that it can be expanded in a Taylor's series, usually about some minimum point. If the series converges rapidly enough, it can be truncated after one or a few terms. If the series takes the form:

$$V(x) = V_0 + (dV/dx)_{x0}(x - x_0) + (1/2)(d^2V/dx^2)_{x0}(x - x_0)^2 + \cdots$$

then, we get force terms of the type

$$F = k_1(x - x_0) + k_2(x - x_0)^2 + \cdots$$

which, in turn, lead to a non-linear differential equation of the form

$$\ddot{x} + \alpha x + \beta x^2 + \gamma x^3 + \cdots = 0.$$

If all but the linear term (αx) are discarded, one has the equation for the simple harmonic oscillator. Thus, the simple harmonic oscillator sometimes becomes the simplest approximation to a more general problem.

In two- and three-dimensional problems, constants of the motion, such as the total energy or a component of angular momentum, may be used to reduce a multiple-dimensioned problem to a one-dimensional differential equation of the type derived above. For example, in the case of periodic motion under a central force, one can convert from a time-dependent differential equation $(\ddot{r}, \dot{r}, \ddot{\theta}, \dot{\theta})$ to an orbit equation relating r and θ (in plane polar coordinates). We use the substitutions:

$$x = 1/r,$$

$$\dot{x} = (dx/d\theta)\dot{\theta} = -(1/r^2)\dot{r}$$

and use the constant of the motion,

$$\ell = mr^2\dot{\theta}.$$

We then have,

$$\dot{r} = -(\ell/m)(dx/d\theta)$$

and

$$\ddot{r} = -(\ell^2 x^2/m^2)(d^2x/d\theta^2)$$

The resulting equation may be a one-dimensional equation that can be solved using the methods described here.

EQUATIONS THAT MAY BE EITHER LINEAR OR NON-LINEAR

Equation: $F(x) = m\ddot{x}$ where F is only a function of position and may be derived from a potential function, $F(x) = -dV/dx$.

Method of solution: Direct integration, separation of variables

Solution: Use the chain rule to write $\ddot{x} = d\dot{x}/dt = (d\dot{x}/dx)(dx/dt) = (dv/dx)v = 1/2d(v^2)/dx$. Then,

$$F(x) = d(1/2mv^2)/dx = -dV/dx.$$

Both sides are perfect differentials, so $T = -V + constant$. This is *energy conservation*, $E = T + V$. From energy conservation, it may be possible to solve for $v = dx/dt = f(x)$ or invert to solve for $x = g(v)$.

If one can write $dx/dt = f(x)$, then,

$$t(x) = \int (1/f(x))dx.$$

In conservative systems, this will take the form,

$$t = (m/2)^{1/2} \int (E - V(x))^{-1/2} dx.$$

Equation: $F(t) = m\ddot{x}$ where F is only a function of time.

Method of solution: Direct integration, separation of variables

Solution:

$$F(t) = m(dv/dt)$$

$$v(t) = (1/m) \int F(t)dt$$

$$x(t) = \int v(t)dt$$

Equation: $F(v) = m\ddot{x}$ where F is only a function of velocity.

Method of solution: Direct integration, separation of variables
Solution: Alternative 1:

$$t = m\int (1/F(v))dv$$

We may then possibly be able to solve for v(t) and integrate again: $x(t) = \int v(t)dt$
Alternative 2:

$$\ddot{x} = (dv/dt) = (dv/dx)(dx/dt) = v(dv/dx)$$

$$x(v) = m\int (v/F(v))dv$$

We may the possibly be able to solve for $v(x) = dx/dt$. Then

$$t(x) = \int (1/v(x))dx$$

We may then possibly be able to invert the expression for $t(x)$ to solve for $x(t)$.

LINEAR DIFFERENTIAL EQUATIONS

The most general *linear* first-order differential equation is:

Equation: $\frac{dy}{dx} + g(x)y = h(x)$.

Method of solution: Integrating factor.
Solution: Observe that

$$\frac{d}{dx}(ye^{\int g(x')dx'}) = (\frac{dy}{dx} + yg(x))e^{\int g(x')dx'}.$$

Thus, multiply the given differential equation by the factor (called the *integrating factor*),

$$e^{\int g(x')dx'},$$

which turns the left-hand-side of the equation into a perfect differential that can be integrated immediately. Then,

$$\frac{d}{dx}(ye^{\int g(x')dx'}) = h(x)e^{\int g(x')dx'}.$$

Integrating both sides, we have,

$$ye^{\int g(x')dx'} = \int h(x)e^{\int g(x')dx'}dx + C,$$

where C is the constant of integration. From this expression, one may solve for $y(x)$ if $g(x)$ can be integrated. Formally,

$$y(x) = e^{-\int g(x')dx'}(\int h(x)e^{\int g(x')dx'}dx + C).$$

However, one should *not* remember the formula itself as a solution. Rather, an equation of this type should be solved by actually carrying out the steps just illustrated,

1. Compute the integrating factor.
2. Multiply the differential equation by the integrating factor.
3. Integrate both sides of the equation, remembering that the left-hand-side is a perfect differential.
4. Solve the integrated equation for $y(x)$.

The first-order equation may occur in mechanics in the following way. If a frictional force proportional to the square of velocity exists, Newton's Second Law may take the form,

$$m\ddot{x} = h(x) - g(x)\dot{x}^2.$$

Observe that

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = m\dot{x}\frac{d\dot{x}}{dx} = m\dot{x}\frac{d\dot{x}}{dt}\frac{dt}{dx} = m\frac{\dot{x}}{\dot{x}}\ddot{x} = m\ddot{x}.$$

Thus,

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = h(x) - g(x)\dot{x}^2.$$

If we now define $y \equiv \dot{x}^2$, we have,

$$\frac{dy}{dx} + \frac{2g(x)}{m}y = \frac{2h(x)}{m}$$

which is of the form of interest. The solution yields velocity as a function of position, but only for problems in which the frictional force is proportion to the square of speed.

The most general *linear* second-order differential equation is:

$$\ddot{x} + f(t)\dot{x} + g(t)x = h(t)$$

Equation: $m\ddot{x} = constant$ (freely falling particle)

Method of solution: Direct integration
Solution: If the constant is -mg (freely falling particle)

$$\dot{x} = \dot{x}_0 - gt$$

$$x = x_0 + \dot{x}_0 t - (1/2)gt^2$$

Equation: $m\ddot{x}+kx=0$ (harmonic oscillator)

Method of solution: Substitution of the form Ae^{mt}
Solution:

$$x=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)=A\cos(\omega_0 t-\alpha)$$

where $\omega_0=\sqrt{(k/m)}$ and c_1 and c_2 or A and α are determined from initial conditions.

Equation: $m\ddot{x}+b\dot{x}=-mg=constant$ (particle falling in a resistive medium)

Method of solution: Integrating factor
Solution:

$$\dot{x}=(mg/b)[e^{-bt/m}(1+b\dot{x}_0/mg)-1]$$

$$x=x_0+(mg/b)[(m/b)(1-e^{-bt/m})(1+b\dot{x}_0/mg)-t]$$

Equation: $m\ddot{x}+b\dot{x}+kx=0$ (particle oscillating in resistive medium)

Method of solution: Substitution of general form, Ae^{mt}
Solution: The general solution is:

$$x=c_1e^{\alpha_1 t}+c_2e^{\alpha_2 t}$$

where

$$\alpha_1=-(b/2m)+\sqrt{(b/2m)^2-k/m}$$

$$\alpha_2=-(b/2m)-\sqrt{(b/2m)^2-k/m}$$

In those cases where α_1 and α_2 are complex numbers, the solution is oscillatory, but damped. It may then be written,

$$x=e^{-(b/2m)t}A\cos(\omega_1 t-\alpha)$$

where

$$\omega_1=\sqrt{k/m}\sqrt{(1-b^2/4mk)}$$

Equation: $m\ddot{x}+kx=F_0\cos(\omega t)$ (driven harmonic oscillator)

Method of solution: Substitute a trial solution of the form $A\cos(\omega t)$
Solution: The solution is the sum of a particular solution plus the solution to the homogeneous equation (right hand side set to 0 , see above). The solution to the homogeneous equation is called the *transient solution*. The transient solution will contain two constants that are determined by the initial conditions.

The particular solution is of the form: $A\cos(\omega t)$. Direct substitution leads to the following:

$$\dot{x} = -[(F_0\omega)/m(\omega_0^2 - \omega^2)]\sin(\omega t)$$

$$x = [(F_0)/m(\omega_0^2 - \omega^2)]\cos(\omega t)$$

where $\omega_0^2 = k/m$.

Equation: $m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega t)$ (damped, driven harmonic oscillator)

(Also, $m\ddot{y} + b\dot{y} + ky = F_0\sin(\omega t)$).

Method of solution: These two equations can be combined and solved simultaneously by defining the complex number $z = x + iy$. By multiplying the second of the two by $i = \sqrt{-1}$ and adding, we form the complex equation,

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\omega t}$$

After solving for z we can recover the solutions for x and y as the real and imaginary parts of z.

The solution is the sum of a particular solution plus the solution to the homogeneous equation (right-hand side set to 0, see above). The solution to the homogeneous equation is called the transient solution. The transient solution will contain two constants that are determined by initial conditions.

Solution: The particular solution is of the form:

$$z = Be^{i(\omega t - \beta)}$$

where,

$$B = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 b^2)}},$$

$$tan\beta = \frac{(\omega b)}{m(\omega_0^2 - \omega^2)},$$

and, $\omega_0^2 = k/m$.

Equation: $m\ddot{x} + b\dot{x} + kx = a_0 + \sum F_n\cos(n\omega t - \alpha_n)$ (damped, periodically driven harmonic oscillator)

Method of solution: The right-hand side of the equation is a Fourier series representation of a periodic driving force. The constants a_0, F_n, and α_n are assumed known from the Fourier series. A complex number $z = x + iy$ is defined and the equation above is replaced with an equation in z. The solution for x will be the real part of the solution for z. The equation becomes:

$$m\ddot{z} + b\dot{z} + kz = a_0 + \sum F_n e^{i(n\omega t - \alpha_n)}$$

The solution is the sum of a particular solution plus the solution to the homogeneous equation (right-hand side set to 0, see above). The solution to the homogeneous equation is called the transient solution. The transient solution will contain two constants that are determined by initial conditions.

We proceed to obtain the particular solution by substituting a trial solution of the form,

$$z = B_0 + \sum B_n e^{i(n\omega t - \gamma_n)}$$

Solution:

$$x = (a_0/k) + \sum B_n \cos(n\omega t - \gamma_n)$$

where,

$$B_n = \frac{F_n}{\sqrt{m^2(\omega_0^2 - n^2\omega^2)^2 + \omega^2 n^2 b^2}}$$

and,

$$tan\beta_n = \frac{bn\omega}{m(\omega_0^2 - n^2\omega^2)}$$

and,

$$\gamma_n = \alpha_n + \beta_n$$

and $\omega_0^2 = k/m$.

The solution is simply a sum of the solutions that would result if each term in the Fourier series expansion of the driving force were present alone! This important *superposition principle* is valid for systems which obey linear differential equations. The superposition principle breaks down for non-linear differential equations.

Equation: $g_{ij}\ddot{q}^j + k_{ij}q^j = 0$ (multi-particle, harmonic oscillation)

Method of solution: If the kinetic energy and potential energy in a conservative, multi-particle system can be expressed as:

$$T = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j$$

$$V = \frac{1}{2} k_{ij} q^i q^j$$

then the above differential equations follow. The equations can also be written in matrix form,

$$G\ddot{Q} + KQ = 0$$

A trial solution of the form $Q = X\cos(\omega t - \alpha)$ can be substituted. Q and X are column vectors. Solutions of the equation,

$$det(K - \omega^2 G) = 0$$

(the *secular equation*) yields the *eigenfrequencies*, ω_i. When these are substituted one by one into the matrix equation,

$$(K - \omega^2 G)X = 0$$

one obtains the individual *eigenvectors*, X_i.

Solution:

$$Q = a_1 X_1 \cos(\omega_1 t - \alpha_1) + \cdots + a_f X_f \cos(\omega_f t - \alpha_f)$$

where $a_i,\ \alpha_i$ are determined by initial conditions.

NONLINEAR DIFFERENTIAL EQUATIONS

Equation: $\ddot{x} + \alpha x + \gamma x^3 = 0$

Method of solution: If,

1. force on the particle is derivable from a potential, V(x)
2. V(x) is symmetric about the Taylor's series point of expansion,
3. the motion is periodic,
4. $\dot{x} = 0$ at $t = 0$

then, the lowest-order, non-linear approximation is $\ddot{x} + \alpha x + \gamma x^3 = 0$.

Since we assume that the solution will be periodic in time, we use symmetry to eliminate terms in the solution of the type $\sin(\omega t)$ and assume a solution of the form,

$$x(t) = b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \cdots$$

We assume that b_3 is smaller than b_1 and discard any terms of higher order. We substitute the form into the differential equation and gather the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$ and separately set them to zero.

Solution:

$$x(t) \approx b_1 \cos(\omega t - \phi) + (\gamma b_1^3/32\alpha)\cos(3\omega t - 3\phi)$$

$$\omega^2 \approx \alpha + (3/4)\gamma b_1^2$$

b_1 and ϕ are determined from initial conditions.

Equation: $\ddot{x}+\alpha x-\beta x^2+\gamma x^3=0$

Method of solution:
If,

1. force on the particle is derivable from a potential, V(x),
2. V(x) is asymmetric about the Taylor's series point of expansion,
3. the motion is periodic,
4. $\dot{x}=0$ at $t=0$,
5. $\beta>0$.

then, the third-order, non-linear approximation is $\ddot{x}+\alpha x-\beta x^2+\gamma x^3=0$

Since we assume that the solution will be periodic in time, we use symmetry to eliminate terms of the type $\sin(n\omega t)$ and assume a solution of the form,

$$x(t)=b_0+b_1\cos(\omega t)+b_2\cos(2\omega t)+b_3\cos(3\omega t)+\ldots$$

We have assumed that the oscillation assumes its maximum amplitude at $t=0$. We assume that $b_1>b_2>b_3$ and discard any terms of higher order. We substitute the form into the differential equation, gather the coefficients of $\cos(\omega t)$, $\cos(2\omega t)$, $\cos(3\omega t)$ and separately set them to zero.

Solution:

$$x(t)\approx(\beta b_1^2/2\alpha)+b_1\cos(\omega t)-(\beta b_1^2/6\alpha)\cos(2\omega t)+((\gamma b_1^3/32\alpha)$$

$$+(\beta^2 b_1^3/48\alpha^2))\cos(3\omega t)$$

where,

$$\omega^2\approx\alpha+(3/4)\gamma b_1^2-(5\beta^2 b_1^2/6\alpha)$$

Equation: $\ddot{x}+\alpha x\pm\gamma x^3=0$

Method of solution:
If,

1. force is derivable from a potential, V(x),
2. and if, the motion is periodic,
3. and if, the potential is symmetric, let $x=Ay(at)$
4. or if, the potential is asymmetric, let $x=B+Ay(at)$

Solution: The resulting equation may possibly be put into one of the forms for which Jacobian elliptic functions are known solutions.

1. $(y')^2=(1-y^2)(1-k^2y^2)$
   $y''+y(1+k^2)-2k^2y^3=0$
   $y=\mathrm{sn}u, y'=(\mathrm{cn}u)(\mathrm{dn}u)$
2. $(y')^2=(1-y^2)(1-k^2+ky^2)$

$y'' + y(1-2k^2) + 2k^2y^3 = 0$
$y = \mathrm{cn}u, y' = (-\mathrm{sn}u)(\mathrm{dn}u)$

3. $(y')^2 = (1-y^2)(y^2-1+k^2)$
$y'' + y(k^2-2) + 2y^3 = 0$
$y = \mathrm{dn}u, y' = -k^2(\mathrm{sn}u)(\mathrm{cn}u)$

4. $(y')^2 = (1+y^2)(1+(1-k^2)y^2)$
$y'' - y(2-k^2) - 2y^3(1-k^2) = 0$
$y = \mathrm{tn}u, y' = (\mathrm{dn}u)/(\mathrm{cn}^2u)$

5. $(y')^2 = (y^2-1)(y^2-k^2)$
$y'' + y(1+k^2) - 2y^3 = 0$
$y = 1/\mathrm{sn}u, y' = -(\mathrm{cn}u)(\mathrm{dn}u)/(\mathrm{sn}^2u)$

6. $(y')^2 = (y^2-1)[(1-k^2)y^2+k^2]$
$y'' + y(1-2k^2) - 2y^3(1-k^2) = 0$
$y = 1/\mathrm{cn}u, y' = (\mathrm{sn}u)(\mathrm{dn}u)/(\mathrm{cn}^2u)$

7. $y' = \sqrt{(1+y^2)^2 - 4k^2y^2}$
$y'' + 2y(2k^2-1) - 2y^3 = 0$
$y = (\mathrm{dn}u)(\mathrm{tn}u)$

8. $(y')^2 = 4y(1-y)(1-k^2y)$
$y'' - 2 + 4y(k^2+1) - 6k^2y^2 = 0$
$y = \mathrm{sn}^2u$

9. $(y')^2 = 4y(1-y)(1-k^2+k^2y)$
$y'' - 2(1-k^2) - 4y(2k^2-1) + 6k^2y^2 = 0$
$y = \mathrm{cn}^2u$

10. $(y')^2 = 4y(1-y)(y-1+k^2)$
$y'' - 2(k^2-1) - 4y(2-k^2) + 6y^2 = 0$
$y = \mathrm{dn}^2u$

11. $(y')^2 = 4y(y-1)(y-k^2)$
$y'' - 2k^2 + 4y(k^2+1) - 6y^2 = 0$
$y = 1/(\mathrm{sn}^2u)$

12. $(y')^2 = 4y(y-1)[(1-k^2)y+k^2]$
$y'' + 2k^2 - 4y(2k^2-1) - 6y^2(1-k^2) = 0$
$y = 1/(\mathrm{cn}^2u)$

13. $(y')^2 = 4y[(1+y)^2 - 4k^2y]$
$y'' - 2 - 8y(1-2k^2) - 6y^2 = 0$
$y = (\mathrm{dn}^2u)(\mathrm{tn}^2u)$

14. $(y')^2 = 4y(1+y)[1+(1-k^2)y]$
$y'' - 2 - 4(2-k^2)y - 6(1-k^2)y^2 = 0$
$y = \mathrm{tn}^2u$

Equation: $\ddot{x} + \omega_0^2 x = f(x,\dot{x})$

Method of solution: (Averaging Method) If the function, f, can be considered a small perturbation on otherwise periodic motion, then we can assume a solution of the form:

$$x(t) = A(t)\cos(\omega_0 t + \phi(t))$$

Assuming that A and ϕ vary only slowly with time, they may be considered constant over one cycle of the motion. When this assumption is made, we can write differential equations for A and ϕ as follows:

$$\dot{A} = -(1/\omega_0)\langle f(x,\dot{x})\sin(\omega_0 t + \phi)\rangle$$

$$\dot{\phi} = -(1/\omega_0)\langle [f(x,\dot{x})\cos(\omega_0 t + \phi)]/A\rangle$$

In performing the integrations, the form for $x(t)$ is substituted, but A and ϕ are considered constant in the time average over one cycle of the motion. The resulting differential equations for A and ϕ are then solved and substituted into the original assumed form to yield the solution.

NUMERICAL SOLUTIONS TO DIFFERENTIAL EQUATIONS

Consider the second-order differential equation,

$$\ddot{x} = f(x,\dot{x},t)$$

We can define a new variable, $z(t)$, such that

$$\dot{x} = dx/dt = z(t)$$

Then, our original equation can be reduced to solving two first-order differential equations:

$$dz/dt = f(x,z,t)$$

$$dx/dt = z$$

Note that the right-hand sides are free of explicit "dotted" variables.

This scheme can be generalized for multi-dimensional systems:

$$dy_i/dt = f_i(y_1,\ldots,y_n,t)$$

The underlying idea for solving the initial value problem is always this: Rewrite the dy's and dt as Δy_i and Δt. Then,

$$\Delta y_i = f_i(y_1,\ldots,y_n,t)\Delta t$$

i.e.

$$y_i(t+\Delta t) = y_i(t) + f_i(y_1,\ldots,y_n,t)\Delta t$$

In the limit of taking Δt to be small (but not too small!), a good approximation to the underlying differential equation is achieved. Literal implementation of this scheme is called *Euler's Method*.

Solving differential equations by numerical methods is something of an art. The idea of the basic Euler method has been modified and improved in a variety of ways to improve accuracy, numerical stability, and speed. Runge-Kutta methods, the Bulirsch-Stoer method, and predictor-corrector methods are some of these variations. None can be said to be "the best." The choice of method depends on the nature of the equation, the nature of the boundary conditions, etc.

There are now a number of computer software packages that solve differential equations by these or other methods. Some packages allow the user to choose among several available methods. In others, the solution occurs in a "black box."

PROBLEMS

1. Computer Project 1 (see Appendix)
2. Computer Project 2 (see Appendix)