A Harmony of the Gospels of **E** and **B**

Ross L. Spencer

The electric field of a point charge and distributions of charge is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\hat{\boldsymbol{\eta}}}{\eta^2} = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')\hat{\boldsymbol{\eta}}d\tau'}{\eta^2} \tag{1}$$

The magnetic field of a moving point charge and distributions of current is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o q\mathbf{v}\times\hat{\boldsymbol{\eta}}}{4\pi\eta^2} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\eta}}d\tau'}{\eta^2} \tag{7}$$

or for currents confined to circuits

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi}\oint \frac{d\mathbf{l}'\times\hat{\boldsymbol{\eta}}}{\eta^2} \tag{8}$$

Electric fields of this form have the special property that their curl is zero, and this coupled with the special nature of the inverse square law gives two vector differential relations for $\mathbf{E}$:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_o} \quad ; \quad \nabla\times\mathbf{E} = 0 \tag{2}$$

Although we have looked hard, we have been unable to find magnetic monopoles, i.e., bare north or south magnetic poles, so magnetic fields have the special property that their divergence is zero, and this coupled with the special nature of the inverse square law gives two vector differential relations for $\mathbf{B}$:

$$\nabla\cdot\mathbf{B} = 0 \quad ; \quad \nabla\times\mathbf{B} = \mu_o\mathbf{J} \tag{9}$$

Because the curl is zero, we can express $\mathbf{E}$ as the gradient of a scalar function V (potential or voltage) which satisfies

$$\mathbf{E} = -\nabla V \quad ; \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{\eta} \quad ; \quad \nabla^2 V = -\frac{\rho}{\epsilon_o} \tag{3}$$

Because the divergence is zero, we can express $\mathbf{B}$ as the curl of a vector potential $\mathbf{A}$ which (if we use the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$) satisfies

$$\mathbf{B} = \nabla\times\mathbf{A} \quad ; \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')d\tau'}{\eta} \quad ; \quad \nabla^2\mathbf{A} = -\mu_o\mathbf{J} \tag{10}$$

In matter, the important concept is the dipole, a distribution of charge in which charges Q and $-Q$ are separated by a distance l. We define the dipole moment of such a distribution to be $\mathbf{p} = Q\mathbf{d}$, where $\mathbf{d}$ points from the negative charge to the positive charge. The electrostatic potential of a dipole is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\eta}}}{\eta^2} \tag{4}$$

In matter, the important concept is the dipole, a closed loop of current I with vector area $\mathbf{a}$ defined by $\mathbf{a} = \frac{1}{2}\oint d\mathbf{l}\times\mathbf{r}$. We define the dipole moment of such a distribution to be $\mathbf{m} = I\mathbf{a}$. In actual matter it is not always clear where this current resides, but careful experiments have shown that particles, atoms, molecules and current loops do indeed have dipole moments, so we just believe the experiments and think about these dipole moments as tiny loops of current only to guide our intuition. The vector potential of a dipole is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\eta}}}{\eta^2} \tag{11}$$

The polarization vector $\mathbf{P}(\mathbf{r})$ inside matter is defined to be

$$\mathbf{P} = \lim_{\Delta V\to 0}\frac{\Delta\mathbf{p}}{\Delta V} \tag{5}$$

and the resulting polarization charge densities are given by

$$\rho_b = -\nabla\cdot\mathbf{P} \quad ; \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \tag{6}$$

These charge densities are used in Eq. (3) to find V and $\mathbf{E}$ both inside and outside of matter.

The magnetization vector $\mathbf{M}(\mathbf{r})$ inside matter is defined to be

$$\mathbf{M} = \lim_{\Delta V\to 0}\frac{\Delta\mathbf{m}}{\Delta V} \tag{12}$$

and the resulting magnetization current densities are given by

$$\mathbf{J}_b = \nabla\times\mathbf{M} \quad ; \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} \tag{13}$$

($\mathbf{K}_b$ is the current per unit length flowing along the magnetized surface.) These current densities are used in Eq. (10) to find $\mathbf{A}$ and $\mathbf{B}$ both inside and outside of matter.

The fact that the polarization charge density is given by the divergence of something makes it possible to simplify problems involving the interaction of electric fields with matter by introducing a new vector field called the electric displacement $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$. This is useful because of what it does to the differential form of Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o} - \nabla \cdot \mathbf{P} \quad \Rightarrow \quad \nabla \cdot \mathbf{D} = \rho \tag{14}$$

where ρ is only the free, or conduction charge; the polarization charge has been absorbed into $\mathbf{D}$. There is a price to be paid for doing this, however: since $\mathbf{P}$ does not necessarily have zero curl then $\mathbf{D}$ doesn't have zero curl either. This means that generally electric fields in matter can't be found through potential theory (solving $\nabla^2 V = 0$), unless the material has certain special properties.

The fact that the magnetization current density is given by the curl of something makes it possible to simplify problems involving the interaction of magnetic fields with matter by introducing a new vector field called the magnetic intensity $\mathbf{H} = \mathbf{B}/\mu_o - \mathbf{M}$. This is useful because of what it does to the differential form of Ampere's law:

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}_{free} + \mu_o \nabla \times \mathbf{M} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}_{free} \tag{17}$$

where $\mathbf{J}$ is only the free, or conduction current; the magnetization current has been absorbed into $\mathbf{H}$. There is a price to be paid for doing this, however: since $\mathbf{M}$ does not necessarily have zero divergence then $\mathbf{H}$ doesn't have zero divergence either. Therefore, we can't get $\mathbf{H}$ from a vector potential like the one we use for $\mathbf{B}$. On the other hand, if there are no conduction currents in the problem, as is the case when all we have is applied magnetic fields and magnetized matter, then $\mathbf{J} = 0$, which means that we have $\nabla \times \mathbf{H} = 0$, and we can find magnetic fields in matter through potential theory (solving $\nabla^2 V = 0$). When we do magnetic problems this way we define a magnetic potential analogous to the electric potential in this way:

$$\mathbf{H} = -\nabla V^* \quad \text{or} \quad \mathbf{B} = -\mu_o \nabla V^* + \mu_o \mathbf{M} \tag{18}$$

For comparison with Eq. (4) the magnetic potential of a point dipole is given by

$$V^*(\mathbf{r}) = \frac{1}{4\pi} \frac{\mathbf{m} \cdot \hat{\boldsymbol{\eta}}}{\eta^2} \tag{19}$$

And since V and V^* correspond, then their sources also correspond, i.e., just as we have polarization charge densities in Eq. (6) we have magnetic "pole densities" that determine V^*:

$$\rho_{bm} = -\nabla \cdot \mathbf{M} \quad ; \quad \sigma_{bm} = \mathbf{M} \cdot \mathbf{n} \tag{20}$$

$$V^*(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{\rho_{bm}(\mathbf{r}')d\tau'}{\eta} \quad + \quad \frac{1}{4\pi} \int_S \frac{\sigma_{bm}(\mathbf{r}')da'}{\eta} \tag{21}$$

One of the special properties that many forms of matter have is that they don't have any intrinsic polarization, but only become polarized when they are acted on by an electric field. And a further special property shared by some forms of matter is that the polarization is simply proportional to the electric field, i.e.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{15}$$

where χ_e is the linear electric susceptibility of the material. This implies a simple linear relation between $\mathbf{D}$ and $\mathbf{E}$ as well, through the permittivity ϵ and the dielectric constant ϵ_r:

$$\mathbf{D} = \epsilon \mathbf{E} \quad ; \quad \epsilon = \epsilon_o(1 + +\chi_e) \quad ; \quad \epsilon_r = \frac{\epsilon}{\epsilon_o} \tag{16}$$

One of the special properties that many forms of matter have is that they don't have any intrinsic magnetization, but only become magnetized when they are acted on by a magnetic field. And a further special property shared by some forms of matter is that the magnetization is simply proportional to the magnetic intensity, i.e.

$$\mathbf{M} = \chi_m \mathbf{H} \tag{22}$$

where χ_m is the linear magnetic susceptibility of the material. This implies a simple linear relation between $\mathbf{H}$ and $\mathbf{B}$ as well, through the permeability μ and the relative permeability constant μ_r:

$$\mathbf{B} = \mu \mathbf{H} \quad ; \quad \mu = \mu_o(1 + \chi_m) \quad ; \quad \mu_r = \frac{\mu}{\mu_o} \tag{23}$$

If χ_e is constant, then electric fields in matter can be found by solving Laplace's equation separately in various regions where χ_e is uniform and then matching the solutions in the different regions to each other by means of boundary conditions.

The boundary conditions satisfied by $\mathbf{E}$ and $\mathbf{D}$ at the interface between medium 1 and medium 2, with surface normal vector $\mathbf{n}$ pointing away from medium 1 and into medium 2, are

$$V_2 - V_1 = 0 \quad ; \quad (\text{same as} \quad E_2^{\parallel} - E_1^{\parallel} = 0) \tag{24}$$

$$D_2^{\perp} - D_1^{\perp} = \sigma_{free} \quad \text{or} \quad \Delta D_n = \sigma_{free} \tag{25}$$

If χ_m is constant, then magnetic fields in matter can be found by solving Laplace's equation separately in various regions where χ_m is uniform and then matching the solutions in the different regions to each other by means of boundary conditions.

The boundary conditions satisfied by $\mathbf{B}$ and $\mathbf{H}$ at the interface between medium 1 and medium 2, with surface normal vector $\hat{\mathbf{n}}$ pointing away from medium 1 and into medium 2, are

$$B_2^{\perp} - B_1^{\perp} = 0 \tag{28}$$

$$(\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel}) = \mathbf{K}_{ext} \times \hat{\mathbf{n}} \quad \text{or} \quad \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_{free} \tag{29}$$

where $\mathbf{K}_{free}$ is the conduction current per unit length on the surface. If you can keep the vectors straight with the right rule, you can also use

$$\Delta H^{\parallel} = K_{free} \tag{30}$$

where $H^{\parallel}$ is the tangential component of $\mathbf{H}$ that is perpendicular to $\mathbf{K}$.

Michael Faraday found that an electric field can be made by a time-varying magnetic field. This field is not of electrostatic type because it has non-zero curl:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{26}$$

The total electric field, electrostatic and electromagnetic combined, is given in terms of the electrostatic potential and the vector potential:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{27}$$

James Clerk Maxwell found that Ampere's law is incomplete, requiring the addition of the effect that magnetic fields can also be made by time-varying electric fields. The corrected equation is

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{31}$$

Since this correction is just an addition to the curl of $\mathbf{B}$ no modification of the connection between $\mathbf{B}$ and the vector potential is required, so we still have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{32}$$

With these connections between $\mathbf{E}$ and $\mathbf{B}$ and each other's time derivatives a harmony is no longer useful. We do not, in fact, have separate electric and magnetic fields given by elliptic partial differential equations. Instead we have a combined electromagnetic field described by Maxwell's system of hyperbolic partial differential equations with characteristic speed equal to the speed of light. The electrostatic and magnetostatic theories of this harmony are only low frequency approximations to the full equations of electromagnetism.